# Generalized Pseudoanalytic Functions with Partial differential Equations

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**Abstract:** We investigate the pseudo-analytic functions of the first kind and the second kind . Some relationship of pseudoanalytic functions with differentiation and integration also considered. The objective of present paper is how to the $(F,G)$-pseudo-annlytic functions of the first (second) kind are reduced pseudo analytic function of I by using general linear homogeneous partial differential equation (with real coefficients).

**Keywords**: Pseudo-analytic functions ,Partial derivatives , General system of linear homogeneous partial differential equations (with real coefficients).

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## 1-Introduction

We discussed in this paper pseudo- analytic functions and their properties which characterized by four properties in the following details . The class $A_D$ of functions $\mathcal{W}(z)$ which are analytic in a fixed domain $D$ then (1) $A_D$ is a real vector space . (2) If $\mathcal{W}(z)$ belongs to $A_D$ and $\mathcal{W}(z_0) = 0$ for some $z_0$ in $D$ the quotient $\mathcal{W}(z)/(z - z_0)$ is continuous at $z_0$. (3) the functions 1 and $i$ belong to $A_D$ (4) $A_D$ is maximal with respect to properties (1),(2),(3) which means the existence of complex derivative $\mathcal{W}'(z)$ at every point of $D$ in order that a function $\mathcal{W}$ should belong to $A_D$ , and property (4) implies that this condition is sufficient. We can obtain pseudoanayltic function by a modifications of the preceding definition [3,4,6]. Partial derivatives are given by subscripts .In particular $2\mathcal{W} = \mathcal{W}_x - i\mathcal{W}_y$ , $2\mathcal{W}_{\bar{z}} = \mathcal{W}_x + i\mathcal{W}_y$. If $\mathcal{W}(z)$ belongs to $A_D(F,G)$ we call $\mathcal{W}$ an $(F,G)$ pseudoanalytic function of the first kind and $\omega$ which correspond $\mathcal{W}$ in the equation $\omega = {}_*\mathcal{W}$ , $\mathcal{W} = \omega^*$ is call pseudoanalytic function of the second kind. The differentiation and integration of pseudoanalytic function are considered by letting two pseudoanalytic functions $F(z)$ and $G(z)$ of class $\mathcal{H}^1$defined in a domain $D_0 \subset E_0$ are said to to from a generating pair in $D_0$ if Im $(\bar{F}G) > 0$. A generating pair $(F_1, G_1)$ is called a successor of $(F,G)$ if $(F,G)$-derivatives are $(F_1, G_1)$-pseudoanalytic ,and $(F_1, G_1)$-pseudo-analytic functions are integerable. Then $(F,G)$ is called a predecessor.Pseudo-analytic Functions and Partial Differential Equations was discussed . In or order to express pseudo-

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analyticity by differential equations we associate with every generating pair $(F, G)$ the function in equations (12),(13),(14) where $\alpha$ and $\beta$ being real-valued.

**Definition 1**

The class $A_D$ of functions $\mathcal{W}(z)$ which are analytic in a fixed domain D can be characterized by four properties (1) $A_D$ is a real vector space . (2) If $\mathcal{W}(z)$ belongs to $A_D$ and $\mathcal{W}(z_0) = 0$ for some $z_0$ in D .Then the quotient $\mathcal{W}(z)/(z - z_0)$ is continuous at $z_0$. (3) The functions 1 and $i$ belongs to $A_D$. (4) $A_D$ is maximal with respect to properties (1),(2),(3). In fact , the first three properties imply that the existence of the complex derivatives $\mathcal{W}'(z)$ at every point of D is necessary in order that a function $\mathcal{W}$ should belong to $A_D$ , and property (4) implies that this condition is sufficient .

Pseudoanalytic functions are obtained by a modification of the preceding definition [3,4,5]. Let $F(z), G(z)$ be two continuous functions defined in domain $D_0$ and satisfying the inequality

$$Im\left\{\overline{F(z)}G(z)\right\} > 0 \qquad (1)$$

(We write functions of $(x, y)$ as function of $z = x + iy$ , without implying analytic .The bar over complex number denotes the complex conjugate ). Partial derivatives are denoted by subscripts . In particular $2\mathcal{W} = \mathcal{W}_x - i\mathcal{W}_y$ , $2\mathcal{W}_{\bar{z}} = \mathcal{W}_x + i\mathcal{W}_y$ . Let $A_D(F, G)$ be the maximal class of the functions . A(complex-valued) function $\mathcal{W}(z)$ belongs to class $\mathcal{C}(class\ \mathcal{H})$ if its continuous (satisfies a Holder condition) . If $\mathcal{W}(z)$ has partial derivatives with respect to $x$ and $y$ of order $n$ , which are of class $\mathcal{C}(of\ class\ \mathcal{H})$ , then $\mathcal{W}(z)$ if of class $\mathcal{C}^n(of\ class\ \mathcal{H}^n)$ . $\mathcal{W}(z)$ is said to have property of pseudoanalytic at $z = \infty$ if $\mathcal{W}\left(\frac{1}{z}\right)$ pseudoanalytic at $z = 0$ . If $\mathcal{W}(z)$ belongs to $A_D(F, G)$ we call $\mathcal{W}$ an $(F, G)$ pseudoanalytic function of the first kind . Its clear that pseudoanalticity of $\mathcal{W}$ in $D$ implies its pseudoanalticity in every subdomain of $D$ .It follows from (1) that every function $\mathcal{W}(z)$ defined in a subdomain of $D$ , whether or not it belongs to $A_D(F, D)$ , admits the unique representation

$$\mathcal{W}(z) = \phi(z)F(z) + \psi(z)G(z) \qquad (2)$$

where the functions $\boldsymbol{\phi}$ , $\boldsymbol{\psi}$ are real -valued . It is convenient to associate with the function $\boldsymbol{\mathcal{W}}$ the function

$$\boldsymbol{\omega(z) = \phi(z) + i\phi(z)} \qquad \mathbf{(3)}$$

If for some $z_0 \in D$ the limit

$$\dot{\mathcal{W}}(z_0) = \lim_{z \to z_0} \frac{\mathcal{W}(z) - \phi(z_0)F(Z) - \psi(z_0)G(z)}{z - z_0} \qquad (4)$$

The correspondence between $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{\omega}$ and one-to-one. We denote it by

$$\omega = {}_*\mathcal{W} \ , \ \mathcal{W} = \omega^* \tag{5}$$

If $\boldsymbol{\mathcal{W}}$ is $(\boldsymbol{F}, \boldsymbol{G})$ pseudoanalytic of the first kind , called $\boldsymbol{\omega}$ pseudoanalytic of the second kind . The class of pseudoanalytic function is closed under addition and multiplication by real constants and uniform convergence .

**2- Differentiation and integration [**2,3,5]

The $(F, G)$ derivative of a function (2) is the defined by the relation

$$\dot{\mathcal{W}}(\mathrm{z}) = \frac{d_{(F,G)}\mathcal{W}(z)}{dz} = \lim_{h\to 0} \frac{[\phi(z+h)-\phi(z)]F(z+h)-[\psi(z+h)-\psi(z)]G(z+h)}{h} \tag{6}$$

Provided that the limit to the right exists and finite .Its to see that a function $\mathcal{W}(z)$ is $(F, G)$ pseudoanalytic of the first kind in a domain $D$ if and only if it possesses an $(F, G)$ derivative at every point of $D$. The generators , which play the role of constants have identically vanishing $(F, G)$ derivatives . It is clear that $(F, G)$ differentiability implies continuity. By letting $h$ in (6) approach zero first through real and then through imaginary values one obtains the existence of the partial derivatives of $\phi$ and $\psi$ , and the relations

$$\phi_{\bar{z}}F + \psi_{\bar{Z}}G = 0, \tag{7}$$

$$\phi_Z F + \psi_Z = \dot{\mathcal{W}} \tag{8}$$

Where the formal differential operators $\frac{\partial}{\partial z}$ , $\frac{\partial}{\partial \bar{z}}$ are defined by the relations

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Those familiar with the theory of elliptic partial differential equations will not be surprised to learn that continuity of the generators is not sufficient to insure the continuity of the $(F, G)$ derivatives .We shall therefore assume that the generators F and G are Holder continuous .Under this hypothesis it can be shown that the $(F, G)$ derivative of an $(F, G)$ pseudoanalytic function is continuous .Thus the class of $(F, G)$ pseudoanalytic functions of the second kind is the class of solutions $(\phi, \psi)$ of the (elliptic) system (7) .It might be worthwhile to state explicitly that equation (8) can not be transformed by a change of dependent and independent variables , into the Cauchy-Riemann system.

**Theorem2**

If $\mathcal{W}(z)$ is pseudo-analytic of the first kind , $\mathcal{W}(z)$ and $\dot{\mathcal{W}}(\mathrm{z})$ are of class $\mathcal{H}^1$. If $\omega(z)$ is pseudo-analytic of the second kind ,$\omega(z)$ is of class $\mathcal{H}^2$

and the mapping $\omega = \omega(z)$ is an interior , sense-preserving ,quasi-conformal transformation .
An isolated singularity $z_0$ of a single-valued pseudo-analytic function $\mathcal{W}(z)$ is called a pole if $\mathcal{W}(z) \sim a(z - z_0)^{-n}, z \to z_0$ , where $a \neq 0$ and $n$ is a positive integer , an essential singularity if $\mathcal{W}(z)$ comes arbitrary close to every complex number as $z \to z_0$ .

**Theorem 3**

An isolated singularity of a single-valued pseudo-analytic function is either essential , or removable , or a pole .
Let $W(z)$ be a continuous function , and Γ a rectifiable curve leading from $z_0$to $z_1$ .The $(F, G)$ –integral of $W$ over Γ defined by the equation:

$$\int_\Gamma W d_{(F,G)} z = F(z_1) Re \int_\Gamma \frac{\bar{G} W dz}{F\bar{G} - \bar{F}G} - G(z_1) Re \int_\Gamma \frac{\bar{F}W}{F\bar{G} - \bar{F}G} \qquad (9)$$

$W$ is called $(F, G)$-integrable in a domain D if its $(F, G)$-integral over a closed curve Γ vanishes whenever Γ is homologous to zero in $D$ .

**Theorem4**

A continuous function is $(F, G)$-integrable if and only if it is an $(F, G)$-derivative;

$$\int_{z_0}^{z_1} \dot{\mathcal{W}} d_{(F,G)} z = \mathcal{W}(z_1) - \phi(z_0) F(z_1) - \psi(z_0) G(z_1) \qquad (10)$$

A generating pair $(F_1, G_1)$ is called a successor of $(F, G)$ if $(F, G)$-derivatives are $(F_1, G_1)$-pseudo-analytic functions are $(F, G)$-integrable .$(F, G)$ is then called a predecessor of $(F_1, G_1)$ .

**Theorem5**

Let $(F, G)$ be a generating pair in $D_0$ . In every bounded domain $D_1$ , $\bar{D}_1 \subset D_0$ possesses a successor and a predecessor .
A sequence of generating pairs $\{(F_\nu, G_\nu)\}, \nu = 0, \pm 1, \pm 2, \ldots$ , is called a generating sequence if $(F_{\nu+1}, G_{\nu+1})$ is a successor of $(F_\nu, G_\nu)$ . A generating pair $(F, G)$ in $D_0$ can be embedded (in every bounded domain $D_1, \bar{D}_I \subset D_0$ ) in generating sequence $\{(F_\nu, G_\nu)\}$ such that $(F_0, G_0) = (F, G)$ .With respect to such a sequence an $(F, G)$-pseudo-analytic function $\mathcal{W}(z)$ has derivatives of all orders:

$$\mathcal{W}^{(0)} = \mathcal{W}, \mathcal{W}^{(n+1)} = d_{(F_n, G_n)} \mathcal{W}^{(n)} / dz, n = 1,2, \ldots \qquad (11)$$

The sequence $\{\mathcal{W}^{(n)}(z)\}$ determines the function $\mathcal{W}(z)$ uniquely .
Tow generating pairs ,$(F, G)$ and $(\tilde{F}, \tilde{G})$ and are called equivalent if

$\tilde{F} = a_{11}G + a_{22}G$ , $\tilde{G} = a_{21}F + a_{22}G$, the $a_{jk}$ being real constants .A generating sequence $\{(F_\nu, G_\nu)\}$ is called periodic if there exists an integer $\mu > 0$ such that $\{(F_{\nu+\mu}, G_{\nu+\mu})\}$ is equivalent to$(F_\nu, G_\nu)$ .

**Remark6**

Since $(F_\nu, G_\nu)$-differentiability is a conformally invariant property , pseudo-analyticity can be defined on a Riemann surface.

**3-Pseudo-analytic Functions and Partial Differential Equations :**

In order to express pseudo-analyticity by differential equations we associate with every generating pair $(F, G)$ the functions

$$a = -(\bar{F}G_{\bar{z}} - F_{\bar{z}}\bar{G})/(F\bar{G} - \bar{F}G), b = (FG_{\bar{z}} - F_{\bar{z}}G)/(F\bar{G} - \bar{F}G) \,, \quad (12)$$

$$A = -(\bar{F}G_z - F_z\bar{G})/(F\bar{G} - \bar{F}G), B = (FG_z - F_zG)/(F\bar{G} - \bar{F}G) \,, \quad (13)$$

$$\alpha - i\beta = 4(\bar{F}/F)(FG_z - F_zG)/(\bar{F}G - F\bar{G}), \quad (14)$$

$\alpha$ *and* β being real-valued.

**Lemma7**

Let $D_0$ and $D_1$ be domains such that $D_1 \subset D_0 \subset E_0$. (a) given two function of class $\mathcal{H}, a(z)$ and $b(z)$, defined in $D_0$There exists a generating pair $(F, G)$ in $D_1$ such that (12) holds. (b) given two real-valued functions of class $\mathcal{H}, \alpha(z)$ and $\beta(z)$ , defined in $D_0$there exists a generating pair $(F, G)$ in $D_1$ such that (14) holds. The statements (a) and (b) remain valid for $D = D_0 = E_0$ if $a, b, \alpha, \beta = 0(|z|^{-2})$ , $z \to \infty$.

**Theorem8**

A function $\mathcal{W}(z) = \phi(z)F(z) + \psi(z)G(z)$ of class $C^1$ is $(F, G)$-pseudo-analytic if and only if

$$\mathcal{W}_{\bar{z}} = a\mathcal{W} + b\overline{\mathcal{W}} \quad (15)$$

i.e., if and only if

$$F\phi_z + G\psi = 0 \quad (16)$$

If this condition is satisfied , then

$$\dot{\mathcal{W}} = \mathcal{W}_z - A\mathcal{W} - B\overline{\mathcal{W}} \equiv F\phi_z + G\psi_z \quad (17)$$

and

$$(\dot{\mathcal{W}})_{\bar{z}} = a\dot{\mathcal{W}} - B\overline{\dot{\mathcal{W}}}. \quad (18)$$

(Note that (18) and part (a) of lemma7 imply the existence of successors.)

Eliminating from (16) function $\phi$ we obtain the equation

$$\psi_{xx} + \psi_{yy} + \alpha\psi_x + \beta\psi_y = 0 \quad (19)$$

where $\alpha$ *and* $\beta$ are given by (14)

**Example 9**

Let $\sigma(z)$ be a positive function of class $\mathcal{H}^1$, and set $(F, G) = (\sigma^{-1/2}, i\sigma^{1/2})$. Then the $(F, G)$-pseudo-analytic function of the first (second) kind are the reduced pseudo-analytic( pseodu-analytic) functions of I .If $\sigma = \sigma_1(x)\sigma_2(y)$, then the function of the second kind are sigma-monogenic see [1].In this case a generating sequence can be obtained by setting

$(F_{2v}, G_{2v}) = (F, G), (F_{2v+1}, G_{2v+1}) = (\tilde{\sigma}^{-1/2}, i\tilde{\sigma}^{1/2})$ where $\tilde{\sigma} = \sigma_2(y)/\sigma_1(x)$.Consider now the general linear homogenous partial differential equation (with real coefficients):

$$A_{11}\psi_{xx} + 2A_{12}\psi_{xy} + A_{22}\psi_{yy} + B_1\psi_x + B_2\psi_y + \Gamma\psi = 0 \quad (20)$$

We assume that the coefficients $A_{jk}$ are of class $\mathcal{H}^1$, the coefficients $B_j$ , of class $\mathcal{H}$ , and the ellipticity condition $(A_{11}A_{22} - A_{12}^2 > 0)$ is satisfied in the domain considered see [6].By introducing new independent variables (20) can be reduced to the form

$$\psi_{xx} + \psi_{yy} + \alpha\psi_x + \beta\psi_y + \gamma\psi = 0 \quad (21)$$

If $\gamma \equiv 0$ , this equation is at the form (19) and its follows from statement (b) of the lemma7 that , for an appropriate choice of $(F, \mathcal{O})$ , the solutions of our equation will be the imaginary parts of $(F, G)$-pseudo-analytic function of the second kind. If $\gamma \not\equiv 0$ ,we consider equation (21) in a domain where it possesses a positive solution $\psi_0$ , and introducing the new unknown function $\tilde{\psi} = \psi/\psi_0$reduce the equation to the form (19) . Consider next the general system of linear homogenous partial differential equations ( with real coefficients).

$$A_{j1}\phi_x + A_{j2}\phi_y + B_{j1}\psi_x + B_{j2}\psi_y + \Gamma_{j1}\phi + \Gamma_{j2}\psi = 0, j = 1,2 . \quad (22)$$

This system is elliptic if the roots of the equation $det(A_{jk} - \lambda B_{jk}) = 0$ Are complex .We assume that this condition is satisfied in the domain considered , and that the coefficients $A_{jk}$ , $B_{jk}$ are of class $\mathcal{H}^1$, the coefficients $\Gamma_{jk}$ are of class $\mathcal{H}$.By introducing new independent variables (22) can be reduced to the form

$$\phi_x = \tau\psi_x + \sigma\psi_y + \gamma_{11}\phi + \gamma_{12}\psi, \phi_y = -\sigma\psi_x + \tau\psi_y + \gamma_{21}\phi + \gamma_{22}\psi \quad (23)$$

where $\sigma$ has constant sign . Without loss of generality we assume that $\sigma > 0$. Set $\gamma^2 = \gamma_{11}^2 + \gamma_{12}^2 + \gamma_{21}^2 + \gamma_{22}^2$. If $\gamma \equiv 0$, equations (23) are equivalent to equation (16) for $F = 1, G = -\tau + i\sigma$. Solutions $(\phi, \psi)$ are then real and imaginary parts of $(F, G)$-pseudo-analytic functions of the second kind .If $\gamma \not\equiv 0$ , we reduce (23) to a system of the same form with $\gamma \equiv 0$ , by introducing new unknown functions $\tilde{\phi}$ , $\tilde{\psi}$ such that $\phi = \phi_1\tilde{\phi} + \phi_2\tilde{\psi}$ , $\psi = \psi_1\phi + \psi_2\psi$ , where $(\phi_1, \psi_1)$ , $(\phi_2, \psi_2)$ are fixed solutions of (23) with $\phi_1\psi_2 - \phi_2\psi_1 > 0$. The existence of such solutions follows from statement (a) of the lemma7 .

On the other hand , we can introduce new unknown functions $u = \phi - \tau\psi$ , $v = \sigma\psi$ then $\mathcal{W} = u + iv$ satisfies an equation (15).It follows from (a) that for appropriate choice of $(F, G)$ , solutions $(u, v)$ are real and imaginary parts of pseudo-analytic functions of the first kind.

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